Using a standard numeraire

Paul Hollingsworth

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1 Purpose

Most options pricing literature came originally from the equities world in which there is a natural choice of numeraire. For example, dynamically replicating an IBM option, to most people, means "buying and selling shares of IBM stock, in exchange for US dollars" (i.e. US dollars are the numeraire). Far fewer would consider it "buying and selling US dollars in exchange for shares of IBM stock"

In FX, and even in equities once you start including Quanto options, there are advantages to being agnostic about the choice of numeraire and hence your underlying, particularly when aggregating and analysing your Delta risk. An example might illustrate better:

1.1 Example

Suppose that I've bought a EURAUD Call, a EURUSD Put and a EURUSD Call. What hedges should I make to offset the spot risk of these three option trades?

It's quite possible that I will actually be completely flat and not need to perform any Delta hedges at all.

To see why, consider each of the spot trades I would do to hedge each option position:

- To hedge the EURAUD Call, I'm going to sell euros and buy Australian dollars

- To hedge the EURUSD Put, I'm going to buy euros and sell US dollars
- To hedge the AUDUSD Call, I'm going to sell Australian dollars and buy US dollars

If you draw it in a table, then it's clear that each currency cancels out:

Hedge	EUR	AUD	USD
EURAUD Call	sell	buy	
EURUSD Put	buy		sell
AUDUSD Call		sell	buy

2 Multi-asset options

Of course, in practise, when you ask the spot desk to do a EURAUD spot trade, they probably cross through the US dollar anyway, because the EURUSD and AUDUSD markets are more liquid.

But this does give an idea for how we could think of FX option greeks so that we hedge and analyze our risk more efficiently. In our EURAUD trade, when we replicate the "EURAUD" option by performing "EURAUD" spot trades, we're *actually* replicating it by performing both EURUSD and AUDUSD trades.

So rather than considering a EURAUD option as a "single underlying" derivative, we could instead consider it to be a two-underlying derivative. It has both a EURUSD underlying and a AUDUSD underlying.

But why even treat US dollars as being special? We could treat all of our derivatives as multi-asset derivatives, replicated by crossing through some fictional currency unit.

We'll call them *Flainian Pobble Beads* or *beads* for short.

We'll say that a EURUSD Call option is really a multi-asset option, with two underlying "prices" that it cares about, the EURbead spot price and the USDbead spot price. By picking a standard numeraire, we allow all of our delta positions to cancel out, while also generically handling a lot of the "result conversion" logic that is currently done as a variety of ad-hoc rules.

3 Notation

We'll use the letters E, U, and A for the spot prices of Euros, US dollars and Australian dollars - but in terms of which currency? The subscript will say which currency.

So:

- E_U is the price of 1 Euro in US dollars
- E_A is the price of 1 Euro in Australian dollars
- A_E is the price of 1 Australian dollar in Euros

We could use b for "beads" but for succinctness let's say that if there's no subscript, we are referring to the price of that currency in 'beads'.

i.e. let:

- E be the price of Euros in beads
- U be the price of US dollars in beads
- A be the price of Australian dollars in beads

4 Spot conversions

These different prices are not independent: If I have x Euros, then I can use these to buy xE beads. Using those beads, I can then buy $x\frac{E}{U}$ US dollars. I've successfully converted x Euros into $x\frac{E}{U}$ US dollars without taking any risk. Therefore, the appropriate arbitrage free 'spot' price of Euros to US dollars must be $\frac{E}{U}$:

$$E_U(E, U) = \frac{E}{U} \tag{1}$$

And in general, if I have any asset V that I can price in some currency Y, then I can relate V_Y - the price of V in currency Y - to the price of V and Y in beads, with:

$$V_Y(V, Y) = \frac{V}{Y} \tag{2}$$

5 Derivatives

5.1 Using US dollars as the numeraire

When someone sells a EURUSD call option, assuming that the underlying EURUSD spot price follows some sort of continuous brownian motion path, I can replicate this option by continually buying and selling Euros for US dollars each time the EURUSD spot price changes.

The typical approach is as follows: Let

- E_U be the price of Euros in terms of US dollars
- $C_U(E_U)$ be the value of the call option ('C' for Call), in US dollars, expressed as a function of the EURUSD spot price

Suppose we've sold a EURUSD option (for now, assuming that we didn't receive a premium), and we've bought Δ_E of the EURUSD spot trade to hedge our risk. What's the right value of Δ so that we are hedged, to first order, against movements in the EURUSD spot price? The value of our portfolio, in US dollars, is:

$$P_U = -C_U(E_U) + \Delta_E E_U \tag{3}$$

We require $\frac{\partial P_U}{\partial E_U} = 0$ so

$$\frac{\partial P_U}{\partial E_U} = -\frac{\partial C_U}{\partial E_U} + \Delta_E \tag{4}$$

$$0 = -\frac{\partial C_U}{\partial E_U} + \Delta_E \tag{5}$$

$$\Delta_E = \frac{\partial C_U}{\partial E_U} \tag{6}$$

5.2 Using beads as the numeraire

So we have, using the existing Black-Scholes formula or whatever, this function C_U that can give us one way to calculate the value of the option in terms of E_U. We can now use (2) to construct a function $V(E, U)$ which can give us the value of the option in terms of E and U directly:

$$\frac{V(E,U)}{U} = C_U(E_U(E,U)) \tag{7}$$

$$V(E,U) = C_U(\frac{E}{U}) \cdot U \tag{8}$$

That is, we can calculate E_U by dividing the Euro and US dollar bead prices, and we can calculate V by multiplying C_U by the US dollar bead price.

5.3 Solving for Δ using beads

Our strategy now is to replicate our option by doing Euro-bead and US-bead spot trades. So there'll be a Δ_E EUR-bead hedge and also a Δ_U USD-bead hedge. P is the value of our portfolio in "beads" now, and is given by:

$$P = -V(E,U) + \Delta_E \cdot E + \Delta_U \cdot U \tag{9}$$

To hedge our Euro exposure, we want $\frac{\partial P}{\partial E} = 0$:

$$\frac{\partial P}{\partial E} = -\frac{\partial V}{\partial E} + \Delta_E \tag{10}$$

$$0 = -\frac{\partial C_U}{\partial E_U}\frac{\partial E_U}{\partial E} \cdot U + \Delta_E \tag{11}$$

$$0 = -\frac{\partial C_U}{\partial E_U}\frac{1}{U} \cdot U + \Delta_E \tag{12}$$

$$0 = -\frac{\partial C_U}{\partial E_U} + \Delta_E \tag{13}$$

$$\Delta_E = \frac{\partial C_U}{\partial E_U} \tag{14}$$

Which, happily, gives us exactly the same value for Δ_E as before (6).
That is:

$$\Delta_E = \frac{\partial V}{\partial E} = \frac{\partial C_U}{\partial E_U} \tag{15}$$

This stands to reason: the combination of our EUR-bead and USD-bead hedges has to *always* amount to the same thing as a simple EURUSD spot trade. Even though "conceptually" we are replicating this EURUSD Call option by buying and selling Euros and US dollars for beads, in physical terms we should be ending up with *pure* EURUSD spot trades - there shouldn't ever be any beads left over.

To hedge our US dollar exposure, we want $\frac{\partial P}{\partial U} = 0$:

$$\frac{\partial P}{\partial U} = -\frac{\partial V}{\partial U} + \Delta_U \tag{16}$$

$$0 = -\frac{\partial V}{\partial U} + \Delta_U \tag{17}$$

$$\Delta_U = \frac{\partial V}{\partial U} \tag{18}$$

But how to calculate $\frac{\partial V}{\partial U}$?

5.4 Linear homogeneity

If we had used as our "numeraire" some sort of "bead cent" (one $\frac{1}{100}$ of a bead), so E and U were 100 times bigger, we'd simply expect V to also be 100 times bigger. Indeed this is the case:

$$V(\alpha E, \alpha U) = C_U(\frac{\alpha E}{\alpha U}) \cdot \alpha U \tag{19}$$

$$= \alpha C_U(\frac{E}{U}) \tag{20}$$

$$= \alpha V(E, U) \tag{21}$$

Another way to say this is to say that $V(E, U)$ is *linearly homogeneous*:

$$\alpha V(E, U) = V(\alpha E, \alpha U) \tag{22}$$

If we differentiate (22) with respect to α we have:

$$V(E, U) = \frac{\partial V}{\partial E} E + \frac{\partial V}{\partial U} U \tag{23}$$

Substituting (18) and (15) we have:

$$V(E, U) = \Delta_E E + \Delta_U U \tag{24}$$

$$\tag{25}$$

The above argument applies for any number of assets:

$$V(S_1, \ldots, S_n) = \sum_{i=1}^{n} \Delta_i \cdot S_i \tag{26}$$

That is, the value of a derivative is equal to the sum of the value of the deltas in each asset. So we can always work out the last delta by looking at all of the other deltas and the actual value of the derivative.

Using (24) we have the following for Δ_U:

$$\Delta_U = \frac{V(E, U) - \Delta_E E}{U} \tag{27}$$

5.5 Premium adjustment

It might appear that we've come stuck here: When replicating our EURUSD option, we never actually wanted to have any beads left over. But that would require that whatever beads we used to buy Δ_E Euros would be financed by selling exactly the right number of US dollars:

$$\Delta_E \cdot E = -\Delta_U \cdot U \tag{28}$$

$$\Delta_U = -\Delta_E \cdot \frac{E}{U} \tag{29}$$

which is certainly not the same as (27).

However, we have forgotten one thing: The premium. If we receive the premium in "beads" then (27) is the correct US dollar delta hedge. But in the actual case of selling a EURUSD Call option, we typically receive and book the premium in US dollars, not in beads!

So suppose that we sold this EURUSD Call option and therefore received $C_U(E_U)$ US dollars as premium. Because we've received the premium in US dollars, we need to adjust the US delta by this amount. Our "Premium adjusted" delta is actually:

$$\Delta_U - C_U(E_U) = \frac{V(E,U) - \Delta_E E}{U} - C_U(E_U) \tag{30}$$

$$= \frac{V(E,U) - \Delta_E E}{U} - \frac{V(E,U)}{U} \tag{31}$$

$$= -\Delta_E \frac{E}{U} \tag{32}$$

Which agrees with (29) as we hoped.

5.6 Gamma

In the conventional view of things, only one gamma is relevant: how much the Euro delta changes for a change in the EURUSD spot price.

However, when looking at things from a multi-currency perspective (e.g. AUD dollars), we are interested in the following sensitivites:

- The sensitivity of the Euro delta with respect to a change in the EUR price
- The sensitivity of the Euro delta with respect to a change in the USD price
- The sensitivity of the USD delta with respect to a change in the EUR price
- The sensitivity of the USD delta with respect to a change in the USD price

Gamma is typically defined so that it has the same dimensions as delta, by multiplying the raw sensitivity by the underlying price.

So the above four sensitivities are expressed as:

$$\Gamma_{EE} = \frac{\partial \Delta_E}{\partial E} E \tag{33}$$

$$\Gamma_{EU} = \frac{\partial \Delta_E}{\partial U} U \tag{34}$$

$$\Gamma_{UE} = \frac{\partial \Delta_U}{\partial E} E \tag{35}$$

$$\Gamma_{UU} = \frac{\partial \Delta_U}{\partial U} U \tag{36}$$

The first two are in Euros, the second two are in US dollars.

Often, for display purposes, the gamma is divided by 100 to give the delta change for a 1% price move.

The above four gammas all share a very useful property: the gammas for a particular currency always sum to zero!

Here is how we reach that conclusion:

Starting with (23):

$$V(E,U) = \frac{\partial V}{\partial E}E + \frac{\partial V}{\partial U}U \tag{37}$$

If we differentiate (37) with respect to E on both sides, we get:

$$\frac{\partial V}{\partial E} = \frac{\partial^2 V}{\partial E^2}E + \frac{\partial V}{\partial E} + \frac{\partial^2 V}{\partial E \partial U}U \tag{38}$$

$$0 = \frac{\partial^2 V}{\partial E^2}E + \frac{\partial^2 V}{\partial E \partial U}U \tag{39}$$

$$\tag{40}$$

If we differentiate (37) with respect to U on both sides, we get:

$$\frac{\partial V}{\partial U} = \frac{\partial^2 V}{\partial U \partial E}E + \frac{\partial^2 V}{\partial U^2}U + \frac{\partial V}{\partial U} \tag{41}$$

$$0 = \frac{\partial^2 V}{\partial E \partial U}E + \frac{\partial^2 V}{\partial U^2}U \tag{42}$$

So, all of the second derivative terms are linked to each other.
Observing that:

$$\Gamma_{EE} = \frac{\partial \Delta_E}{\partial E}E = \frac{\partial^2 V}{\partial E^2}E \tag{43}$$

$$\Gamma_{EU} = \frac{\partial \Delta_E}{\partial U}U = \frac{\partial^2 V}{\partial U \partial E}U \tag{44}$$

$$\Gamma_{UE} = \frac{\partial \Delta_U}{\partial E}E = \frac{\partial^2 V}{\partial U \partial E}E \tag{45}$$

$$\Gamma_{UU} = \frac{\partial \Delta_U}{\partial U}U = \frac{\partial^2 V}{\partial U^2}U \tag{46}$$

We can substitute these into (39) and (42) to get:

$$0 = \Gamma_{EE} + \Gamma_{EU} \tag{47}$$

$$0 = \Gamma_{UE} + \Gamma_{UU} \tag{48}$$

The above argument generalizes easily for more than two currencies.

That is: in any risk system that actually shows cross gammas, we expect all of the gammas for each currency to sum to 0.

In the two currency case, because the cross gammas are related to each other by:

$$E \cdot \Gamma_{EU} = U \cdot \Gamma_{UE} \tag{49}$$

we can also then link Γ_{EE} and Γ_{UU} directly:

$$U.\Gamma_{UU} = E \cdot \Gamma_{EE} \tag{50}$$

$$\Gamma_{UU} = \frac{E}{U}\Gamma_{EE} \tag{51}$$

$$\Gamma_{UU} = E_U \cdot \Gamma_{EE} \tag{52}$$

That is, the two are related to each other by the EURUSD fx rate, E_U.

So, for the two currency case, all four gammas can be obtained once you have just one of them.

6 Conclusion

All of the above arguments work equally well for derivatives that depend on more than two currencies, and for more complicated payoffs than European style options. Every position in our book can be thought of as some function V of a bunch of asset prices, each denominated in "beads". By aggregating our delta risk in this form, we'll be sure to net off everything that should be netted off, presenting a unified view of our delta risk.

The following algorithm is what I recommend:

- Rephrase the valuation function as a function V in "beads", whose input asset prices are also all in "beads"
- To get the non-premium adjusted delta for each asset, just differentiate V with respect to that asset's "bead" price
- Don't forget that you'll want to subtract the premium from one of the deltas (or remember to account for premium when aggregating deltas).
- If V depends on N assets, you only need to differentiate V $N-1$ times - the last delta can be obtained due to the fact that V must and will always be linearly homogeneous

To view gamma in a 'multi-asset' world we need to worry about cross gammas. But this isn't as bad as it sounds because for a single currency pair, we can compute the other gammas once one of them is known.

This is because the gammas (including cross gammas) for a given delta currency must sum to zero.