Rho and Delta

Paul Hollingsworth

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1 Introduction

In this article I'll describe the basic idea of viewing a derivative in terms of a portfolio of 'zero coupon bonds'. Viewing the derivative this way provides a nice way to hedge and analyze both the spot risk (Delta) and interest rate risk (Rho) to determine if we are properly hedged.

Just as a 'spot trade' is the exchange of one asset for another at a given price, a 'forward trade' can be viewed as an exchange of one zero coupon bond for another zero coupon bond at a given ratio (strike).

By executing 'forward' trades instead of spot trades, both the delta and rho risk of a portfolio can be hedged.

2 Zero coupon bond

A zero coupon bond is a pure cash flow of 1 unit of currency that will be paid out at some time t years from now. In t years you will receive 1 unit of currency. What's the arbitrage driven price of such a cash flow?

Suppose I have a Euro bank account which pays a continuously compounded interest rate of r_e and I deposit x_0 Euros in it. So for any arbitrarily small time step dt, I have $x(t)$ Euros in the account, and the amount of interest I get, dx, is determined by:

$$dx = r_e x(t) dt \tag{1}$$

$$\frac{dx}{x(t)} = r_e dt \tag{2}$$

Integrating both sides and noting that $x(0) = x_0$, we get:

$$\int \frac{dx}{x(t)} = \int r_e dt \tag{3}$$

$$\ln x(t) = r_e t + C \tag{4}$$

$$x(t) = x_0 exp(-r_e t) \tag{5}$$

So if I deposit x_0 Euros then after time t this will have grown to $x_0 \exp(r_e t)$ Euros in the bank account.

I can use this bank account to replicate a zero coupon bond. We want to determine the fair value of a zero coupon bond paying a notional of 1 Euro at some time t in the future. I can find the quantity x_0 that will ensure that I have 1 Euro at time t in the future with:

$$x_0 \exp(r_e t) = 1 \tag{6}$$

$$x_0 = \exp(-r_e t) \tag{7}$$

Therefore, the arbitrage free price of a zero coupon bond that pays 1 at time t is $\exp(-r_e t)$ Euros:

$$B_E(r_e, t) = \exp(-r_e t) \tag{8}$$

'$\exp(-r_t t)$' is also called the "Euro discount factor" for time t. Any Euro cash flow that is payable at time t can be converted into a value "today" by multiplying by this discount factor. The step of converting a future cash flow into its value today implied by current interest rates is called "present valuing":

$$D(r_e, t) = \exp(-r_e t) \tag{9}$$

$$x \text{ Euros payable at time } t = xD(r_e, t) \tag{10}$$

It's helpful to think of everything in terms of some fictional currency so that equations work out symmetrically regardless of the numeraire. If E denotes the value of 1 Euro in "beads", then the value of a zero coupon bond in "beads" is:

$$B(E, r_e, t) = E \exp(-r_e t) \tag{11}$$

Another name for a "zero coupon bond" is a "zero strike call".

3 FX forward

An FX forward is an agreement to exchange one asset in return for another at a fixed strike, X, at some point t in the future. For example, a EURUSD FX forward struck at X is an agreement to exchange X US dollars for 1 Euro at a point t in the future.

But this is the identical position to *owing* (i.e. being short) X US zero coupon bonds and *owning* (i.e. being long) 1 Euro zero coupon bond.

Let's denote the Euro bond price in "beads" as E_B, and the US dollar bond price in "beads" as U_B. We have:

$$E_B(E, r_e, t) = E \exp(-r_e t) \tag{12}$$
$$U_B(U, r_u, t) = U \exp(-r_u t) \tag{13}$$

A forward agreement struck at X is just the sum of being long the Euro bond and short X of the US dollar bond:

$$F(E_B, U_B) = E_B - XU_B \tag{14}$$
$$= E \exp(-r_e t) - XU \exp(-r_u t) \tag{15}$$

This determines the fair value of a FX forward contract given the strike X. However, typically things work in reverse to this - the strike X is negotiated such that the forward contract has a value of 0 when it is initiated:

$$F(E_B, U_B) = E_B - XU_B \tag{16}$$
$$0 = E_B - XU_B \tag{17}$$
$$X = \frac{E_B}{U_B} \tag{18}$$
$$= \frac{E \exp(r_e t)}{U \exp(r_u t)} \tag{19}$$
$$= \frac{E}{U} \exp((r_e - r_u)t) \tag{20}$$
$$= E_U \exp((r_e - r_u)t) \tag{21}$$

or alternatively, the spot price times the ratio of the discount factors

$$\tag{22}$$
$$= E_U \frac{D(r_e, t)}{D(r_u, t)} \tag{23}$$

People sometimes treat $r_e - r_u$ as a single "interest rate" associated with the EURUSD currency pair.

In reality, X is observed in the market, from which one can imply $r_e - r_u$. Given some basic set of discount factors, all other discount factors can be implied by the forward differentials.

4 European Call under Black Scholes

A EURUSD Call option strike at strike X is the right, but not the obligation, to exchange X US dollars for 1 Euro. Its value in US dollars, E_U, is given by the formula

$$C_U(E_U, r_e, r_u, X, t, \sigma) = E_U \exp(-r_e t)\Phi(d_1) - X \exp(-r_u t)\Phi(d_2) \tag{24}$$

Where

$$d_1 = \frac{ln\frac{E_U}{X} + (r_u - r_e + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \tag{25}$$

$$d_2 = d_1 - \sigma\sqrt{t} \tag{26}$$

We can re-express this with all asset values in "beads" to get the value C, in "beads":

$$C(E, U, r_e, r_u, X, t, \sigma) = E\exp(-r_e t)\Phi(d_1) - XU\exp(-r_u t)\Phi(d_2) \tag{27}$$

with

$$d_1 = \frac{ln\frac{E\exp(-r_e t)}{U\exp(-r_u t)} + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \tag{28}$$

$$d_2 = d_1 - \sigma\sqrt{t} \tag{29}$$

But notice that the terms involving the asset prices are always together with their discount factors. We can write the Black Scholes formula as a function of bond prices. A EURUSD Call option has the following value in terms of the Euro and US dollar bond:

$$C(E_B, U_B, X, t, \sigma) = E_B\Phi(d_1) - XU_B\Phi(d_2) \tag{30}$$

with

$$d_1 = \frac{ln\frac{E_B}{U_B} + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \tag{31}$$

$$d_2 = d_1 - \sigma\sqrt{t} \tag{32}$$

5 Rho (ρ)

Suppose I have some instrument whose value, V is a function of the instanteously compounded interest rate for asset A:

$$V = f(r_A) \tag{33}$$

We define "rho" for asset A, ρ_A, to be the sensitivity of V in units of A, V_A, to a change in r_A.

$$\rho_A = \frac{\partial V_A}{\partial r_A} \tag{34}$$

$$= \frac{1}{A}\frac{\partial V}{\partial r_A} \tag{35}$$

So for example, a EURUSD Call option's Euro 'rho', ρ_E, is the sensitivity of the Euro value of the option, C_E, to the Euro interest rate, r_e. Similarly, the US dollar Rho, ρ_U, is the sensitivity of the US dollar value of the option, C_U, to the US dollar interest rate, r_u:

$$\rho_E = \frac{\partial C_E}{\partial r_e} \tag{36}$$

$$= \frac{1}{E}\frac{\partial C}{\partial r_e} \tag{37}$$

$$\rho_U = \frac{\partial C_U}{\partial r_u} \tag{38}$$

$$= \frac{1}{U}\frac{\partial C}{\partial r_u} \tag{39}$$

6 Relationship between 'Rho' and 'Delta'

There's a useful relationship between the 'rho' and the 'delta' of a bond:

$$E_B = E\exp(-r_e t) \tag{40}$$

$$\Delta_E = \frac{\partial E_B}{\partial E} \tag{41}$$

$$= \exp(-r_e t) \tag{42}$$

$$\rho_E = \frac{1}{E}\frac{\partial E_B}{\partial r_e} \tag{43}$$

$$= \frac{1}{E}\cdot -tE\exp(-r_e t) \tag{44}$$

$$= -t\exp(-r_e t) \tag{45}$$

$$\rho_E = -t\Delta_E \tag{46}$$

So for any bond, we know that $\rho = -t\Delta$.

In fact, this also holds true for any derivative whose value is purely expressed as a function of one or more bond prices.

i.e. suppose we have some derivative whose valuation function V is purely expressed as a function E_B - so other than its dependence on E_B it has no dependence on E or r_e separately.

$$V = f(E_B) \tag{47}$$

$$\Delta_E = \frac{\partial V}{\partial E} \tag{48}$$

$$= \frac{\partial V}{\partial E_B}\frac{\partial E_B}{\partial E} \tag{49}$$

$$= \frac{\partial V}{\partial E_B}\exp(-r_e t) \tag{50}$$

$$\rho_E = \frac{1}{E}\frac{\partial V}{\partial r_e} \tag{51}$$

$$= \frac{1}{E}\frac{\partial V}{\partial E_B}\frac{\partial E_B}{\partial r_e} \tag{52}$$

$$= \frac{1}{E}\frac{\partial V}{\partial E_B} - tE\exp(-r_e t) \tag{53}$$

$$= \frac{\partial V}{\partial E_B}\exp(-r_e t) \tag{54}$$

$$\rho_E = -t\Delta_E \tag{55}$$

In words: the chain rule says that both the rho and the delta for the derivative are a linear multiplier of the 'rho' and 'delta' of the underlying bond. This multiplier, $\frac{\partial V}{\partial E_B}$, is the same for both the 'rho' and the 'delta', therefore the linear relationship holds for the derivative because it holds for the bond.

7 Time bucketing

This relationship also gives us an algorithm for 'time bucketing' the delta risk.

Suppose we have some EURUSD exotic option (e.g. a barrier) that we know has interest rate sensitivity to dates t_1 and t_2.

So we'll assume that there are Euro and US dollar zero coupon bonds for those maturities:

$$E_{B_1} = E\exp(-r_{E_1}t_1) \tag{56}$$

$$E_{B_2} = E\exp(-r_{E_2}t_2) \tag{57}$$

$$U_{B_1} = U\exp(-r_{U_1}t_1) \tag{58}$$

$$U_{B_2} = U\exp(-r_{U_2}t_2) \tag{59}$$

Where E_{B_1} is the value, in 'beads', of a zero coupon Euro bond expiring at time t_1 and so on.

So we're saying that the derivative, V, is not just a function of E and U, the relative values of Euros to US dollars, but is also a function of the zero coupon bond prices for maturities t_1, t_2 for both assets:

$$V = f(E, U, E_{B_1}, E_{B_2}, U_{B_1}, U_{B_2}) \tag{60}$$

So we're saying that we are wishing to model the derivative as if we had this function, but in reality what we have is a function, M, that depends on E, B, $\mathbf{Y_E}$, the Euro yield curve, and $\mathbf{Y_U}$, the US dollar yield curve:

$$V = M = f(E, U, \mathbf{Y_E}, \mathbf{Y_U}) \tag{61}$$

To time bucket the Euro delta, we can use the following approach.

We use finite differencing to determine the ρ to each maturity by bumping $\mathbf{Y_E}$ and observing its effect on M:

$$\rho_{E_1} \approx \frac{1}{E} \cdot \frac{M(\mathbf{Y_E}+\delta) - M(\mathbf{Y_E})}{\delta} \tag{62}$$

Where $\mathbf{Y_E} + \delta$ means 'the Euro yield curve perturbed such that only r_{E_1} is different'.

We do this for each of the dates that we care about. That is, we time-bucket the rho exposure.

Once we have the Rho for each date we care about, we use $\rho = -t\Delta$ to determine the amount of the total 'Delta', $\frac{\partial M}{\partial E}$, that belongs at each maturity.

The remaining delta can remain as a 'spot' delta, $(\frac{\partial V}{\partial E})$, because $t = 0$ for this case.

What we end up with is the portfolio of zero coupon bonds, and raw cash, which most closely replicates the derivative's interest rate and spot risk.

This can then be used to inform which forward trades we do in order to hedge.

8 Summary

The classic "Black Scholes" argument for hedging a derivative assumes that interest rates stay constant.

In practice, interest rates can change over time and therefore there is interest rate risk that needs to be hedged as well as spot risk.

If you hedge a derivative by trading a portfolio of bonds, both the spot risk (Delta) and the interest rate risk (Rho) can be hedged simultaneously.

A forward trade is equivalent to exchanging two zero coupon bonds, therefore an equivalent statement is to say that a derivative can be properly hedged for both delta and rho risk by executing forward trades of the appropriate maturity.

Determining the portfolio of 'zero coupon bonds' which most accurately manages the delta and rho risk of a portfolio is also called 'time bucketing of delta'.

A way to do this is to determine the 'Rho' risk to different maturities and then determine the implied delta risk via the relationship $\rho = -t\Delta$.